The Name of the Game

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Abstract

In the past centuries, there was no problem or fault in misleading names for same definitions or similar methods due to lack of access to information such as journals and internet.

Unfortunately, one of the serious challenges in Mathematics is the originality of the new definitions and methods presented in recent decades. With a little research and accuracy in some of them, it can be shown that some of these definitions and methods are not new. I will illustrate in this talk that there are similarities to the classical methods or some other exiting methods.

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Contents

1 Fractional Derivative and Fractional Integral

Mathematics is **the art of giving things misleading names**. [15]
The beautiful and at first look mysterious - name *the fractional calculus* is just one of the those misnomers which are the essence of mathematics.
For example, we know such names as *natural numbers* $\mathbb{N}$ and *real numbers* $\mathbb{R}$. We use them very often: let us think for a moment about these names. The notion of a *natural number* is a natural abstraction, but is the *number* itself *natural*? The notion of a *real number* is a generalization of the notion of a natural number. The word real emphasizes that we pretend that they reflect real quantities. The *real numbers* do reflect real quantities. but this cannot change the fact that they do not exist.

Everything is in order in mathematical analysis, and the notion of a *real number* makes it easier, but if one wants to compute something, he immediately discovers for himself that there is no place for *real numbers* in the *real world*; nowadays, computations are performed mostly on digital computers, which can work only with finite sets of finite fractions, which serve as approximations to unreal *real numbers*.

1.1 Brief History

Fractional calculus deals with generalizations of the ordinary differentiation and integration to non-integer (real/complex) orders. This subject is as old as the calculus of differentiation and goes back to the times of Leibniz, Gauss, and Newton.

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1},\ f''(x) = n(n-1)x^{n-2},\ \cdots,\ f^{(n)}(x) = n!$$

The first reported attempts to generalize derivatives to fractional order is contained in the correspondence of Leibniz (1695) with L'Hôpital [9]. In a letter to L'Hôpital in 1695 Leibniz raised the following question: ”Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?” The story goes that L'Hôpital was somewhat curious about that question and replied by another question to Leibniz. ”What if the order will be 1/2?” Leibniz in a letter dated September 30, 1695 replied: ”It will lead to a paradox, from which one day useful consequences will be drawn.” In these

Figure 1: Guillaume l'Hôpital (1661-1704) and Gottfried Wilhelm Leibniz (1646-1716)

words fractional calculus was born. Following L'Hôpital's and Leibniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [11]. Mathematicians like Abel, Gauss, Fourier, Grünwald, Riemann, Liouville, Weyl, Letnikov *etc.* made major contributions to the subject of fractional calculus. Abel was the first person who applied fractional derivative for solving a generalized version of the tautochrone problem. He solved the following integral equation for $\alpha = -\frac{1}{2}$,

$$k = \int_0^x (x-t)^\alpha f(t)dt, \tag{1.1}$$

where $f(t)$ is unknown. For determining f, Abel wrote the right hand side of the Eq.(1.1) as $\sqrt{\pi}\left[\frac{d^{-1/2}f(x)}{dx^{-1/2}}\right]$ and applied $\frac{d^{-1/2}f(x)}{dx^{-1/2}}$ on both sides of Eq.(1.1) to obtain:

$$\frac{d^{1/2}k}{dx^{1/2}} = \sqrt{\pi}f(x),$$

as the fractional operators (under suitable conditions on f) have the property:

$$\frac{d^{1/2}}{dx^{1/2}}\left[\frac{d^{-1/2}f(x)}{dx^{-1/2}}\right] = \frac{d^0 f(x)}{dx^0} = f(x).$$

This is a remarkable achievement of Abel, which gave impetus to the development of fractional calculus. He further applied the theory of fractional calculus and applied it successfully to problems in potential theory [9]. Many definitions that fit the concept of a non-integer order integral or derivative exist in the literature. The most famous of these definitions in the world of fractional calculus are the Riemann-Liouville, Grunwald-Letnikov, Caputo, Nishimoto, Modified Riemann-Liouville and more recently Caputo-Fabrizio definitions.

This subject has gained importance and popularity during the past three decades or so, due mainly to its demonstrated applications in numerous and seemingly diverse fields of science and engineering [6, 26]. It indeed provides several potentially useful tools for solving differential and integral equations. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found.

1.2 Applications

There are many fields of applications where we can use the fractional calculus, such as: Viscoelasticity, Control theory, Heat conduction, Electricity, Mechanics, Chaos, Fractals and so on. Recently Maxwell's equations have been generalized using fractional derivatives to better understand multipole moments [4, 5]. Riewe [16, 17] has formulated Lagrangian and Hamiltonian mechanics involving fractional derivatives which leads to equations of motion with non-conservative forces such as friction. Further, Agrawal [1] has developed calculus of variations including fractional order derivatives. Many structures found in nature can be modelled by fractals [8, 19]. Fractals are often so irregular that methods of ordinary calculus are either inapplicable or ineffective. Fractional calculus has proved effective in formulating processes involving fractal structures and phenomenon [3, 12, 24]. Fractional calculus is also being applied to statistics [13, 14].
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A mathematical model for simulation of a water table profile between two parallel subsurface drains using fractional derivatives

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An analytical model for simulating the water table profile between two parallel subsurfaces was derived by solving the linear fractional Boussinesq equation for one-dimensional transient flow toward subsurface drains. The developed model is a generalization of GloverDumms mathematical model. The proposed model is applicable for both homogeneous and heterogeneous soils.

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Regular Article

Derivation of a fractional Boussinesq equation for modelling unconfined groundwater

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In the classical Boussinesq equation, the scale effects are shown as scale-dependent changes in hydraulic characteristics (e.g. hydraulic conductivity, specific yield). Unlike the classical Boussinesq equation, due to the non-locality property of fractional derivatives, the hydraulic characteristics of the fractional Boussinesq equation are constant and scale-invariant. The second distinction is that the fractional Boussinesq equation has two various fractional orders of differentiation with respect to x and y that indicate the degree of heterogeneity in the x and y directions, respectively.

1.3 Basic Definitions

1.3.1 The Mittag-Leffler function

The Mittag-Leffler function is an important function that finds widespread use in the field of fractional calculus. It is a generalization of the exponential function. Just as the exponential function naturally arises in the solution of integer order differential equations, the Mittag-Leffler function plays analogous role in the solution of non-integer order differential equations. The standard definition of the Mittag-Leffler function [26] is

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + 1)}, \quad \alpha > 0. \tag{1.2}$$

The exponential function corresponds to $\alpha = 1$ in the expression (1.2). In figure (2) illustrates the Mittag-Leffler function for $\alpha = 1,\ 1.5,\ 2,\ 5$.

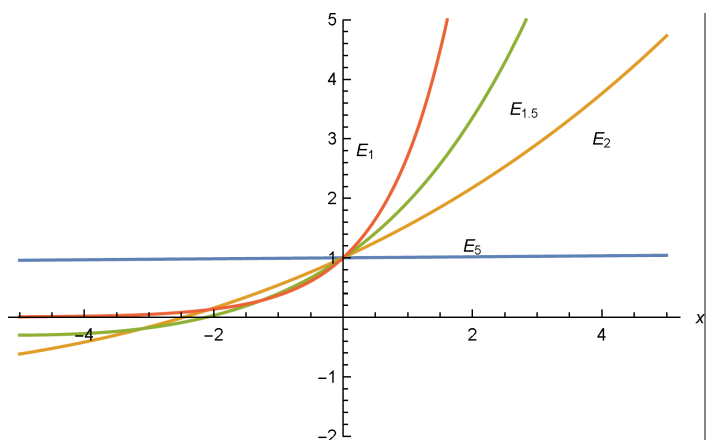


Figure 2: Mittag-Leffler Function

A two-parameter Mittag-Leffler function is defined by the series expansion [26]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)}, \quad (\alpha > 0,\ \beta > 0). \tag{1.3}$$

Multivariate Mittag-Leffler function [7, 26] is defined below.

$$E_{(\alpha_1,\cdots,\alpha_n),\beta}(z_1, z_2, \cdots, z_{n+1}) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_{n+1}=k \\ l_j \geq 0}} (k; l_1, \cdots, l_{n+1}) \left[\frac{\prod_{j=1}^{n+1} z_j^{l}}{\Gamma(\beta + \sum_{j=1}^{n+1} \beta_j l_j)} \right].$$

1.3.2 Leibniz Rule for differentiating an integral

For differentiation of the integral $\int_{f(x)}^{h(x)} G(x,t)\,dt$ with respect to x, we apply the useful Leibniz rule given by [46]:

$$\frac{d}{dx}\int_{f(x)}^{h(x)} G(x,t)\,dt = G(x,h(x))\frac{dh(x)}{dx} - G(x,f(x))\frac{df(x)}{dx} + \int_{f(x)}^{h(x)} \frac{\partial G(x,t)}{\partial x}\,dt, \qquad (1.4)$$

where $G(x,t)$ and $\frac{\partial G(x,t)}{\partial x}$ are continuous functions in the domain D in the xt-plane that contains the rectangular region $\mathbb{R}, a \leq x \leq b,\ t_0 \leq t \leq t_1$ and the limits of integration $f(x)$ and $h(x)$ are defined functions having continuous derivative for $a < x < b$.

1.4 Grünwald-Letnikov's 'Differintegral'

Let $f(x) \in C[a,b]$. The first order derivative of the function $f(x)$ is defined as

$$f'(x) = \frac{df}{dx} = D^{(1)}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ if it exists.} \qquad (1.5)$$

Applying this definition twice we can find the second-order derivative

$$f''(x) = \lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}.$$

By induction,

$$f^{(n)}(x) = D^{(n)}f(x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{m=0}^{n}(-1)^r\binom{n}{m}f(x-mh), \qquad (1.6)$$

where $\binom{n}{m} = \frac{\Gamma(n+1)}{m!\,\Gamma(n-m+1)}$. Expression (1.6) can be generalized for non-integer values of n. For positive values of n it represents fractional derivative and for negative values it represents fractional integral. Let us now consider the following expression where α is non-integer.

$${}_t^G D_a^{\alpha} f(x) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{m=0}^{\frac{t-a}{h}}(-1)^m\binom{\alpha}{m}f(x-mh), \quad \alpha > 0, \qquad (1.7)$$

where $\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha-m+1)}$ and t and a are the upper and lower limits of differentiate-tion, respectively. Equation (1.7) is called Grünwald-Letnikov (GL) derivative.

Example 1.1

$$\begin{aligned}
{}^{G}D_a^{\alpha} e^{ax} &= \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\alpha} (-1)^n \binom{\alpha}{m} e^{a(x+(\alpha-n)h)} = e^{ax} \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\alpha} (-1)^n \binom{\alpha}{m} (e^{ah})^{\alpha-n} \\
&= e^{ax} \lim_{h\to 0} h^{-\alpha} (e^{ah}-1)^{\alpha} = a^{\alpha}\, e^{ax}.
\end{aligned}$$

The GL derivative can be extended to include negative values of α, using generalized binomial coefficients. Generalized binomial coefficient is defined as:

$$\binom{-\alpha}{m} = (-1)^m \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)\, m!}, \qquad \alpha > 0. \tag{1.8}$$

Using (1.8) we can now extend (1.7) for negative orders, *i.e.*

$${}^{G}D_a^{-\alpha} f(x) = \lim_{h\to\ 0} h^{\alpha} \sum_{m=0}^{\frac{t-a}{h}} (-1)^m \frac{\Gamma(\alpha+m)}{m!\,\Gamma(\alpha)} f(x-mh), \tag{1.9}$$

which is called GL fractional integral.

Example 1.2

Let us consider several particular cases.

For $p = 1$ we have:

$$\begin{aligned}
D^{-1} f(x) &= \lim_{\substack{n\to\infty \\ h\to 0}} h \sum_{m=0}^{n} \frac{\Gamma(m+1)}{m!\Gamma(1)} f(x-mh) = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{m=0}^{n} h f(x-mh) \\
&= \lim_{\substack{n\to\infty \\ h\to 0}} \int_0^{nh} f(x-t)dt = \lim_{\substack{n\to\infty \\ h\to 0}} \int_{x-nh}^{x} f(t)dt = \int_a^x f(t)dt, \quad h = \frac{x-a}{n},
\end{aligned}$$

Using induction, we get the following general expression:

$${}^{G}D_a^{-p} f(t) = \lim_{\substack{n\to\infty \\ h\to 0}} h^p \sum_{m=0}^{n} \frac{\Gamma(m+p)}{m!\Gamma(p)} f(t-mh) = \frac{1}{(p-1)!} \int_a^t (t-\tau)^{p-1} f(\tau)d\tau. \tag{1.10}$$

1.5 Riemann-Liouville Fractional Integral/Derivative

Riemann-Liouville fractional integral operator is a direct generalization of the Cauchy's formula for an n-fold integral.

Definition 1.1 *Cauchy's formula for an n-fold integral is given by [44]*

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} f(x_n)dx_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt. \tag{1.11}$$

Formula (2.36) is very valuable and enables the calculations to work. Furthermore, this formula will be used to convert the initial value problems to Volterra integral equations.

Definition 1.2 *If $f(x) \in C[a,b]$ and $\alpha > 0$ then*

$$
\begin{aligned}
I^{\alpha}_{a^+} f(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a, \\
I^{\alpha}_{b^-} f(x) &:= \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x < b,
\end{aligned}
\tag{1.12}
$$

are called as the left sided and the right sided Riemann-Liouville fractional integral of order α, respectively.

The definition of Riemann-Liouville fractional derivative of order α is motivated by Abel's integral equation for any $\alpha \in (0,\ 1)$. Consider the integral equation:

$$
f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0. \tag{1.13}
$$

By solving Eq.(1.13) we get

$$
\phi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt. \tag{1.14}
$$

Thus solution of Eq.(1.13) is given by (1.14) for $\alpha \in (0,1)$. As Eq.(1.13) is the integral of order α, it is natural to define the inversion (1.14) as derivative of order α. The definition of the Riemann-Liouville fractional derivative for arbitrary value of $\alpha > 0$, is

Definition 1.3 *Let $n - 1 < \alpha \le n$ then the left sided and right sided Riemann-Liouville fractional derivatives of order α are defined as:*

$$
\begin{aligned}
\mathbb{D}^{\alpha}_{a^+} f(x) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha+1-n}} dt = D^n I^{n-\alpha}_{a^+} f(x), \quad x > a, \\
\mathbb{D}^{\alpha}_{b^-} f(x) &:= \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{f(t)}{(x-t)^{\alpha+1-n}} dt = D^n I^{n-\alpha}_{b^-} f(x), \quad x < b,
\end{aligned}
\tag{1.15}
$$

respectively, whenever the RHS exist.

In further discussion, unless mentioned otherwise, we denote $\mathbb{D}^{\alpha}_{a^+} f(x)$ by $\mathbb{D}^{\alpha}_{a} f(x)$ and $I^{\alpha}_{a^+} f(x)$ by $I^{\alpha}_{a} f(x)$, respectively. Also $\mathbb{D}^{\alpha} f(x)$ and $I^{\alpha} f(x)$, refer to $\mathbb{D}^{\alpha}_{0^+} f(x)$ and $I^{\alpha}_{0^+} f(x)$, respectively.

Property 1.1 *For any $f \in C[a,b]$ the R-L fractional integral satisfies*

$$I_a^{\alpha} I_a^{\beta} f(x) = I_a^{\beta} I_a^{\alpha} f(x) = I_a^{\alpha+\beta} f(x). \tag{1.16}$$

$$\mathbb{D}_a^{\alpha} I_a^{\alpha} f(x) = f(x). \tag{1.17}$$

$$I_a^{\alpha} \mathbb{D}_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} [\mathbb{D}^{\alpha-k} f(x)]_{x=a} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}. \tag{1.18}$$

In particular if $\left[\mathbb{D}^{\alpha-k} f(x)\right]_{x=a} = 0$ for $k = 0, 1, \cdots, n-1$, we have $I_a^{\alpha} D_a^{\alpha} f(x) = f(x)$. Below we present some fractional integrals.

$f(x)$	$I_a^{\alpha} f(x)$	Specifications
C	$\frac{c}{\Gamma(\alpha+1)}(x-a)^{\alpha}$	$\alpha \in \mathbb{R},\ a \in \mathbb{R}$
$(x-a)^{\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}$	$\alpha > 0,\ Re(\beta) > -1$
e^{bx}	$b^{-\alpha} e^{bx}$	$a = -\infty,\ \alpha > 0,\ Re(b) > 0$

1.6 Caputo Fractional Derivative

Although the R-L definition of fractional derivatives seems to play an important role in the development of fractional calculus, several authors including Caputo (1967, 1969) realized that the R-L definition needs revision because the applied problems in viscoelaticity, solid mechanics and in rheology require physically interpretable initial conditions such as $f(0),\ f'(0),\ f''(0)$.

Caputo reformulated the 'classic' definition of the Riemann-Liouville fractional deriva-

Figure 3: Michael Caputo

tive in order to use integer order initial conditions to solve his fractional order differential equations [26].

Definition 1.4 *Let $f \in C^n[a,b]$ and $n-1 < \alpha < n$ then*

$$D_a^{\alpha} f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(\alpha - n)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{(\alpha-n+1)}} dt, \quad a < x < b, \tag{1.19}$$

is called as Caputo fractional derivative.

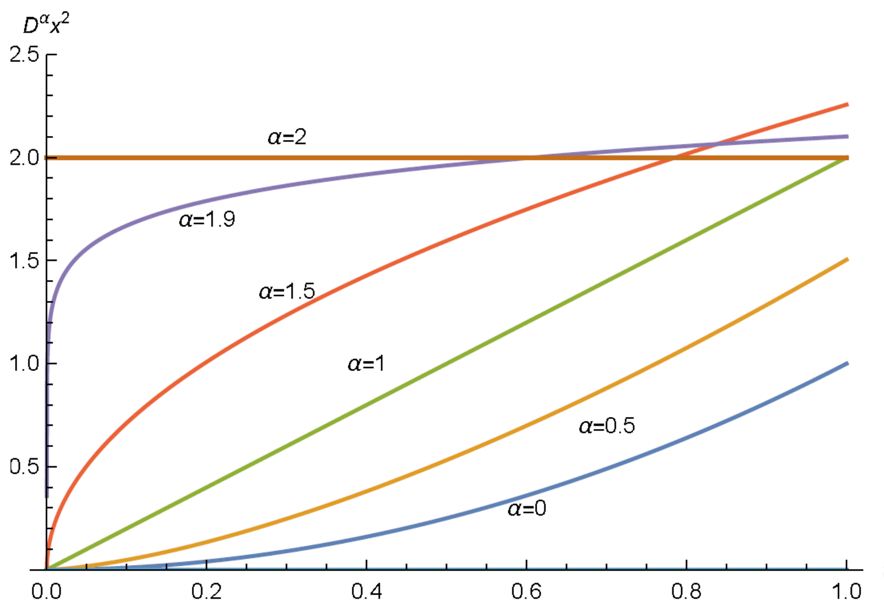
Properties

$$(i) \qquad D_a^{\alpha} C = 0, \quad C \text{ is a constant.} \tag{1.20}$$

$$(ii) \qquad \lim_{\alpha \to n} {}^c D_a^{\alpha} f(x) = f^{(n)}(x). \tag{1.21}$$

$$(iii) \qquad D_a^{\alpha} D^m f(x) = D_a^{\alpha+m} f(x), \quad n-1 < \alpha < n \ , m \in \mathbb{N}. \tag{1.22}$$

Thus for $\alpha \longrightarrow n$ the Caputo fractional derivative becomes the ordinary nth order derivative of the function.



Relation between Riemann-Liouville and Caputo fractional derivatives

Theorem 1.1 *Let $f \in C^n[a,b]$ and $n-1 < \alpha < n$. then the R-L and the Caputo fractional derivatives are connected by the relation*

$$\mathbb{D}_a^{\alpha} f(x) = D_a^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{\Gamma(1+k-\alpha)} (x-a)^{k-\alpha}. \tag{1.23}$$

From above theorem we get the follow result:

i) If $\alpha = n \in \mathbb{N}$, then $\mathbb{D}_a^\alpha f(x) = D_a^\alpha f(x) = D^n f(x)$.

ii) If $f^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$, then $\mathbb{D}_a^\alpha f(x) = D_a^\alpha f(x)$.

iii) If $0 < \alpha < 1$, then $\mathbb{D}_a^\alpha f(x) = D_a^\alpha f(x) + \dfrac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}$.

Theorem 1.2 *Let $f \in C^n[a, b]$ and $n - 1 < \alpha < n$ then*

$$I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!}(x-a)^k, \;\; x \geq a. \tag{1.24}$$

1.7 Nishimoto Fractional Differintegration

Nishimoto defines the fractional derivative as follows [10, 11]

Definition 1.5 *let $C = \{C_-, C_+\}, D = \{D_-, D_+\}$ where C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$,C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$,D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+ and $f = f(z)$ be a regular function in D. Then the fractional differintegration of any arbitrary order ν for $f(z)$ is defined as follows,*

$$\begin{aligned} f_\nu(z) &= \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{1+\nu}} d\xi & (1.25) \\ (f)_{-m} &= \lim_{\nu \to -m} (f)_\nu, \quad (m \in \mathbb{Z}^+), & (1.26) \end{aligned}$$

where

$$-\pi \leq arg(\xi - z) \leq \pi \quad for\, C_-, \quad 0 \leq arg(\xi - z) \leq 2pi for\, C_+, \tag{1.27}$$

$$\xi \neq z, \qquad z \in \mathbb{C}, \nu \in \mathbb{R}, \tag{1.28}$$

$f_\nu(z)$ is the fractional differ-integration of arbitrary order ν with respect to z when $\nu > 0$ or the fractional integral of order $-\nu$ when $\nu < 0$ if $|(f)_\nu| < \infty$.

1.8 Modified Riemann-Liouville derivative

Jumarie defined modified Riemann-Liouville derivatives of order α and some important properties as following [23]:

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)], & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)], & 0 < \alpha < 1, \\ [f^{(\alpha-n)}(x)]^{(n)}, & n \leq \alpha < n+1, \ \ n \geq 1. \end{cases} \tag{1.29}$$

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{1.30}$$

$$D_x^\alpha [f(x)g(x)] = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \tag{1.31}$$

$$D_x^\alpha f[g(x)] = f'_g[g(x)] D_x^\alpha g(x) = D_g^\alpha f[g(x)](g'_x)^\alpha, \tag{1.32}$$

Counterexample

Let $f(x) = x^2$ and $g(x) = x^2$ then in view of (1.30) and (1.31) we have

$$\begin{aligned} D_x^\alpha [f(x)g(x)] = D_x^\alpha [x^4] &= \frac{\Gamma(5)}{\Gamma(5-\alpha)} x^{4-\alpha} \\ g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x) = 2x^2 D_x^\alpha x^2 = 2x^2 \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} &= \frac{2\Gamma(3)}{\Gamma(3-\alpha)} x^{4-\alpha} \end{aligned}$$

It means $D_x^\alpha [f(x)g(x)] \neq g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x)$.

1.9 Caputo-Fabrizio fractional derivative

The new fractional derivative was defined as follow: [**?**, 21, 22].

Definition 1.6 *Let* $0 < \kappa < 1$. *The Caputo-Fabrizio fractional derivative of a function* $\xi(x) \in H^1(a,b), b > a$ *is defined as follows [**?**]*

$$^{CF}D_t^\kappa \xi(t) = \frac{M(\kappa)}{(1-\kappa)} \int_a^t \xi'(\tau) \exp(-\frac{\kappa(t-\tau)}{1-\kappa}) d\tau. \tag{1.33}$$

In the above definition, $M(\kappa)$ is a normalization function. $^{CF}D^\kappa \xi(t)$ is zero when $\xi(x)$ is constant, as usual Caputo derivative but for $t = \tau$ there is no singularity in kernel like the usual Caputo derivative.

Later Losada and Nieto [21] present the anti-derivative the above derivative as follows:

Definition 1.7 *[21] Let $\kappa \in (0,1)$. The the Caputo-Fabrizio fractional integral associated is defined as:*

$$^{CF}I_t^{\kappa}\xi(t) = \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\xi(t) + \frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \xi(\tau)d\tau. \tag{1.34}$$

Remark 1.1 *Losada and Nieto calculated the normalized function $M(\kappa)$ by using to the above definition [21] which is*

$$M(\kappa) = \frac{2}{2-\kappa}, \qquad 0 < \kappa < 1.$$

Hence, they proposed that the definition (1.6) can be rewritten as [21]

$$^{CF}D_t^{\kappa}\xi(t) = \frac{1}{1-\kappa}\int_a^t \xi'(\tau)\exp(-\frac{\kappa(t-\tau)}{1-\kappa})d\tau.$$

Theorem 1.3 *Let $0 < \kappa < 1$, then for every $\xi(t)$ we have*

$$^{CF}I^{\kappa}\left(^{NC}D_t^{\kappa}\xi(t)\right) = \xi(t) - \xi(0). \tag{1.35}$$

1.10 Atangana-Baleanu Derivative

Definition 1.8 *Let $0 < \alpha < 1$. The Atangana-Baleanu fractional derivative of a function $f(x) \in H^1(a,b), b > a$ in Caputo sense is defined as follows*

$$^{ABC}D_t^{\alpha}f(t) = \frac{M(\alpha)}{(1-\alpha)}\int_a^t f'(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau. \tag{1.36}$$

Definition 1.9 *Let $0 < \alpha < 1$. The Atangana-Baleanu fractional derivative of a function $f(x) \in H^1(a,b), b > a$ and not necessary differentiable then, the definition of the AtanganaBaleanu fractional derivative in RiemannLiouville sense is defined as follows*

$$^{ABR}D_t^{\alpha}f(t) = \frac{M(\alpha)}{(1-\alpha)}\frac{d}{dt}\int_a^t f(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau. \tag{1.37}$$

After reviewing the definition of two differential operators which have been recently introduced by Caputo and Fabrizio and, separately, by Atangana and Baleanu, we present an argument for which these two integro-differential operators can be understood as simple realizations of a much broader class of fractional operators, i.e. the theory of Prabhakar fractional integrals. Furthermore, we also provide a series expansion of the Prabhakar

Figure 4: Andrea Giusti

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ORIGINAL PAPER

A comment on some new definitions of fractional derivative

Andrea Giusti

integral in terms of Riemann–Liouville integrals of variable order. Then, by using this last result we finally argue that the operator introduced by Caputo and Fabrizio cannot be regarded as fractional. Besides, we also observe that the one suggested by Atangana and Baleanu is indeed fractional, but it is ultimately related to the ordinary RiemannLiouville and Caputo fractional operators.

1.11 Local fractional calculus

The idea of local fractional calculus, which was first suggested by Kolwankar and Gangal [24] based on the Riemann-Liouville fractional derivative [25, 26], was employed to deal with non-differentiable problems in science and engineering [28]. Yang et al. [27–31] presented the logical generalizations of the definitions to the subject of local derivative on fractals.

Definition 1.10 *Let $f(x) \in C_\alpha(a, b)$. Local fractional derivative of $f(x)$ of order α at*

$x = x_0$ is defined as, [27–31],

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad 0 < \alpha \leq 1, \tag{1.38}$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

Suppose that for any point $x \in (a, b)$ there exists $f^{(\alpha)}(x) = \frac{d^\alpha f(x)}{dx^\alpha} = D^\alpha_x f(x)$ In this case, $D^\alpha_x(a, b)$ is called a α-local fractional derivative set and $f(x) \in D^\alpha_x(a, b)$. Local fractional derivative meets the following simple rules, [27–31],

$$D^\alpha_x c = 0, \quad D^\alpha_x[cf] = cD^\alpha_x f, \quad D^\alpha_x x^\beta = \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \alpha)} x^{\beta-\alpha}, \;\; \beta \geq \alpha > 0, \tag{1.39}$$

and the following simple chain rules

$$D^{k\alpha}_x f(x) = \overbrace{D^\alpha_x D^\alpha_x \cdots D^\alpha_x}^{k\ times} f(x), \quad D^\alpha_x[(f\ o\ g)(x)] = \left(\frac{dg}{dx}\right)^\alpha D^\alpha_x f(g(x)). \tag{1.40}$$

Definition 1.11 *Let $f(x) \in C_\alpha(a, b)$. Local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is given by, [27–31],*

$$_aI^{(\alpha)}_b f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(x)(dx)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta x\to 0} \sum_{j=0}^{N-1} f(x_j)(\Delta x_j)^\alpha, \quad 0 < \alpha \leq 1, \tag{1.41}$$

where $\Delta x_j = x_{j+1} - x_j$, $\Delta x = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_j\}$, and $[x_j, x_{j+1}]$, $j = 0, 1, \cdots, N - 1$, $x_0 = a$, $x_N = b$, is a partition of the interval $[a, b]$.

Suppose that for any point $x \in (a, b)$ there exists $_aI^\alpha_x f(x)$. In this case, $I^\alpha_x(a, b)$ is called a α-local fractional integral set and $f(x) \in I^\alpha_x(a, b)$.

proved that the conformable fractional derivative, the alternative and M-fractional derivatives, the local fractional derivative of Kolwankar and Gangal, the CaputoFabrizio fractional derivatives with exponential kernels cannot be considered as fractional derivatives of non-integer orders and all results obtained for this type of operators can be derived by using differential operators of integer orders. This means that all results obtained for these operators can be derived by using the differential operators with integer orders. Therefore, the proposed operators do not give anything new at least in spaces of differentiable functions except for the change of notations.

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Short communication

No nonlocality. No fractional derivative

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2 New Methods or New Names

In the past centuries, there was no problem or fault in misleading naming for a definition or method because of no communication. For example we use the following triangle to remember the coefficient of $(a + b)^n$.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

which is called **Pascal's Triangle**, **Khayym's Triangle** or **Yang Hui's Triangle**.
The triangle was studied by B. Pascal, although it had been described centuries earlier by **Chinese mathematician Yanghui** (about 500 years earlier, in fact) and **the Persian astronomer-poet Omar Khayym**.

But nowadays we saw still people use different name for a same technique or made a little changes in existing method and give new name!!
I want to discuss about few of them which is related to my research and publications:

During these four decades several analytical and semi analytical methods introduced for solving nonlinear differential equations such as:

Figure 5: Blaise Pascal (1623-1662), Yang Hui(1238 - 1298), Omar Khayyam (1048-1131)

i) The Adomian decomposition method (ADM)

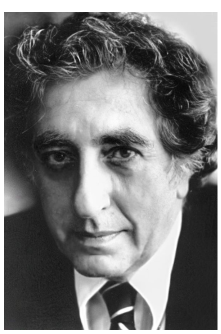
ii) The Homotopy perturbation method (HPM)

iii) The Variational iteration method (VIM)

iv) The Homotopy analysis method (HAM)

v) The Differential transform method (DTM)

vi) The modified variational iteration method (MVIM) or
The variational homotopy perturbation method (VHPM)



Prof. G. Adomian

Prof. J. H. He

Prof. Sh. J. Liao

Also

i) The tanh Method

ii) $(\frac{G'}{G})$-expansion Method

iii) The Simplest equation Method

iv) $\sin - \cos$ method

2.1 The SAM, VIM, ADM, HPM, VHPM

In this section, we briefly recall the ADM, the HPM and the VHPM for solving nonlinear differential equations.

The successive approximations method (SAM) is one of the well know classical methods for solving integral equations [44]. It is also called the Picard iteration method in the literature. In fact, this method provides a scheme that one can use for solving integral equations or initial value problems. One starts by finding successive approximations to the solution by writing an initial guess, called the zeroth approximation, which is any selective real-valued function that one uses in a recurrence relation to determine the other approximations [44].

The variational iteration method was first proposed by He [20] and has been used by many authors over a number of years **G. Adomian** introduced a decomposition method which is called after that as the Adomian Decomposition Method

J. H. He developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems.

Consider the following nonlinear differential equation:

$$L[u(t)] + R[u(t)] + N[u(t)] = g(t), \quad t > 0, \tag{2.1}$$

subject to the initial conditions,

$$u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \cdots, m-1. \tag{2.2}$$

$L = \dfrac{d^m}{dt^m}, m \in \boldsymbol{N}$ is a linear operator, $\qquad R[u(t)]$ (residual linear term),

We want to obtain a solution u of (2.1) in Hilbert Space.

If (2.1) has not a unique solution, then these methods give only a solution among other possible solutions.

2.1.1 Successive approximations method for solving Eq. (2.1)

The successive approximations method considers the approximate solution of an integral equation a sequence usually converging to the accurate solution [44]. For solving equation (2.1) using SAM we apply $L^{-1}[.]$, which is

$$L^{-1}[.] = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}[.]d\tau, \tag{2.3}$$

on both sides of (2.1) so that we have

$$u(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u(t)]) - L^{-1}(N[u(t)]). \tag{2.4}$$

The SAM consists of representing the solution of (2.4) as a sequence

$$\{u_n(x)\}_{n=0}^{\infty}. \tag{2.5}$$

The method introduces the recurrence relation

$$u_{n+1}(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u_n(t)]) - L^{-1}(N[u_n(t)]), \tag{2.6}$$

where the zeroth approximation $u_0(x)$ is an arbitrary real function. Several successive approximations $u_n, n \geq 1$ will be determined as

$$\begin{aligned}
u_1(t) &= \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u_0(t)]) - L^{-1}(N[u_0(t)]),\\
u_2(t) &= \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u_1(t)]) - L^{-1}(N[u_1(t)]),\\
&\vdots\\
u_{n+1}(t) &= \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u_n(t)]) - L^{-1}(N[u_n(t)]),
\end{aligned} \tag{2.7}$$

and the solution computed as:

$$u(x) = \lim_{n\longrightarrow\infty} u_n(x) \tag{2.8}$$

The SAM is very simple in its principles. The difficulties consist in proving the convergence of the introduced series. For convergence of this method we refer the reader to [38].

2.1.2 Variation Iteration Method for solving Eq. (2.1)

We now briefly describe the VIM. For solving (2.1) using VIM, first according to the He's variational iteration method [20], we construct a correction functional for (2.1) as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left\{Lu_n(\tau) + R\tilde{u}_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)d\tau\right\}, \quad n \geq 0 \tag{2.9}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via variational theory and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. Here, we apply restricted variations to nonlinear term Nu.In this case we can easily determine the Lagrange multiplier.

Taking the variation of (2.9) with respect to the independent variable u_n we find

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(\tau)(Lu_n(\tau))d\tau.$$

Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\tau)$. In general, when $Ru_n(\tau) = 0$ or we consider $Ru_n(\tau)$ as a nonlinear term we have [45]

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)} \tag{2.10}$$

Substituting (2.10) in (2.9), where the restrictions should be omitted, yields the approximate solution

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}\left\{Lu_n(\tau) + Ru_n(\tau) + Nu_n(\tau) - g(\tau)\right\}d\tau. \tag{2.11}$$

The success of the method depends on the proper selection of the initial approximation u_0. Like SAM the solution of (2.1) will be calculate as

$$u(x) = \lim_{n\longrightarrow\infty} u_n(x) \tag{2.12}$$

This means that the correction functional (2.9) gives several approximations, which in turn means that the exact solution can be obtained as the limit of the resulting successive approximations.

2.1.3 Comparison between SAM and VIM for solving Eq. (2.1)

In this section we prove that the SAM and the VIM give same solution for solving nonlinear differential equations and that these methods are in fact equivalent.

Theorem 2.1 *The VIM method for solving equation (2.3) is the SAM with the Lagrange multiplier λ given by (2.10).*

Proof. First we prove that

$$\int_0^t \lambda(\tau)[.]d\tau = -L^{-1}[.], \tag{2.13}$$

where $L^{-1}[.]$ is defined with (2.3).
In view of (2.56), we have

$$\begin{aligned}\int_0^t \lambda(\tau)[.]d\tau &= \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}[.]d\tau \qquad (2.14)\\ &= \int_0^t \frac{(-1)^m}{(m-1)!}(-1)^{m-1}(t-\tau)^{m-1}[.]d\tau \\ &= -\int_0^t \frac{1}{(m-1)!}(t-\tau)^{m-1}[.]d\tau = -L^{-1}[.], \quad m = 0, 1, 2, \ldots.\end{aligned}$$

Substituting (2.14) in (2.11), we have

$$u_{n+1} = u_n(t) - L^{-1}\left\{Lu_n(\tau) + Ru_n(\tau) + Nu_n(\tau) - g(\tau)\right\}. \tag{2.15}$$

Using (2.54) we can rewrite (2.15) as

$$\begin{aligned}u_{n+1} &= u_n(t) - u_n(t) + \sum_{k=0}^{m-1} u^{(k)}(0)\frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u_n(t)])L^{-1}(N[u_n(t)]) \\ &= \sum_{k=0}^{m-1} u^{(k)}(0)\frac{t^k}{k!} + L^{-1}(g(\tau)) - L^{-1}(R[u_n(t)] - L^{-1}(N[u_n(t)]) \qquad (2.16)\end{aligned}$$

The proof of Theorem 2.1 is completed.

2.1.4 The ADM for solving Eq. (2.1)

The method was developed from the 1970s to the 1990s by George Adomian, chair of the Center for Applied Mathematics at the University of Georgia.
For solving equation (2.1) using the ADM , we apply $L^{-1}[.] = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1}[.]d\tau$ on both side of (2.1). Thus

$$u(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[u(t)]) - L^{-1}(N[u(t)]), \quad t > 0 \tag{2.17}$$

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A comparison between the variational iteration method and the successive approximations method

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ABSTRACT

In this paper we investigate and compare the variational iteration method and the successive approximations method for solving a class of nonlinear differential equations. We prove that these two methods are equivalent for solving these types of equations.

The ADM consists the solution of (2.55) as an infinite series

$$u(x) = \sum_{i=0}^{\infty} u_i(x), \tag{2.18}$$

and $N(u(x))$ is also decomposed as

$$N(u(x)) = \sum_{i=0}^{\infty} A_i, \tag{2.19}$$

where $A_n, n = 1, 2, 3, \cdots$ are called the Adomian polynomials which are calculated by [33, 34, 44]

$$A_n = \frac{1}{n!}\frac{d^n}{dp^n}[N(\sum_{i=0}^{n} u_i p^i)]|_{p=0}. \tag{2.20}$$

Here p is a parameter introduced for convenience. Upon substituting (2.18) and (2.19) into (2.55) yields

$$\sum_{i=0}^{\infty} u_i(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)) - L^{-1}(R[\sum_{i=0}^{\infty} u_i(t)]) - L^{-1}(\sum_{i=0}^{\infty} A_i). \tag{2.21}$$

In view of the convergence of the series in(2.21), the components of series (2.18) are computed by following formula:

$$u_0 = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + L^{-1}(g(t)), \tag{2.22}$$

$$u_{n+1} = -L^{-1}(R[u_n]) - L^{-1}(A_n), \quad n = 0, 1, 2, \cdots$$

When the independent variable(time)is unbounded,the series solution (2.18) will diverge from the true solution at larger values of time. This is where the discretization of time axis makes itself indispensable. An estimate of local error over a particular time interval is given by $\text{Error}_l = \sum_{i=n+1}^{\infty} u_i \equiv O(\delta h^n)$. The global error order is one integral order less than the corresponding local error order. It is $\text{Error}_l \equiv O(\delta h^{n-1})$. So it may achieve more accurate solution and get higher rate convergence by increasing the number of series terms [47, 48]. For convergence of this method we refer to [33, 34, 47].

2.1.5 The HPM for solving Eq. (2.1)

The Homotopy perturbation method, first proposed by J.H. He in 1998. The HPM was developed by combining two techniques: the standard homotopy and the perturbation. For solving (2.1) according to the He's HPM [36], we first construct a homotopy as

$$\mathcal{H}(v; p) = (1 - p)[L(v) - L(u_0)] + p[L(v) + R(v) + N(v) - g(t)] = 0, \tag{2.23}$$

or

$$\mathcal{H}(v; p) = L(v) - L(u_0) + pL(u_0) + p[R(v) + N(v) - g(t)] = 0, \tag{2.24}$$

where $p \in [0, 1]$ and u_0 is an initial guess of (2.1), which satisfies (2.54). In the HPM, a power series of p

$$v = v_0 + v_1 p + v_2 p^2 + \cdots , \tag{2.25}$$

is considered as the solution of(2.24). Substituting $p = 1$ in (2.23), it gives our original equation (2.1). Also when p tends to 1 in (2.25) we have

$$u(t) = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{2.26}$$

Like the ADM, $N(v)$ is decomposed as

$$N(v) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \cdots , \tag{2.27}$$

where H_n is calculated as

$$H_n(v_0, v_1, v_2, \cdots , v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} (N(\sum_{i=0}^{n} p^i v_i))|_{p=0}, \quad n = 0, 1, 2, \cdots \tag{2.28}$$

it called by few authors as **He's polynomials!**
Substituting (2.25) and (2.27) into (2.23) or (2.24) and arranging it according to the powers of p, we have

$$\begin{aligned}
p^0 :& \quad L(v_0) - L(u_0) = 0, \qquad (2.29)\\
p^1 :& \quad L(v_1) + L(u_0) + R(v_0) + H_0 - g(t) = 0,\\
p^2 :& \quad L(v_2) + R(v_1) + H_1 = 0, \quad {v_2}^{(k)}(0) = 0, \quad k = 0, 1, 2, \cdots, m-1\\
\vdots&\\
p^n :& \quad L(v_n) + R(v_{n-1}) + H_{n-1} = 0, \quad {v_n}^{(k)}(0) = 0, \quad n = 2, 3, \cdots
\end{aligned}$$

By solving the above equations (2.29), we obtain the components $v_i, i = 0, 1, 2, \cdots$ of (2.25). For convergence of this method we refer to [36].
We have proved that He's polynomials is only the Adomian polynomials [39].

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Research Article

A Comparison between Adomian's Polynomials and He's Polynomials for Nonlinear Functional Equations

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2.1.6 The VHPM for solving Eq. (2.1)

Now we briefly describe an alternative approach of VIM which is called MVIM [43] or VHPM [41, 42]. This method is proposed by the coupling of the VIM and the HPM.

The modified variational iteration method (MVIM) is same with exciting method which is called variational homotopy perturbation method (VHPM) [35, 41, 42] For solving (2.1) using the VHPM, first according to the VIM [37,38], a correction functional for (2.1) is constructed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{Lu_n(\tau) + R\,\tilde{u_n}(\tau) + N\,\tilde{u_n}(\tau) - g(\tau)\}d\tau, \quad n \geq 0, \tag{2.30}$$

where λ is a general Lagrangian multiplier, which can be identified optimally via variational theory.

In general [45] we have

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}. \tag{2.31}$$

After finding the value of λ, unlike the VIM and similar to the HPM, we decompose the solution of (2.1) as a following series

$$v = v_0 + v_1 p + v_2 p^2 + \cdots, \tag{2.32}$$

substituting $p = 1$ in (2.32), yields the approximate solution of (2.30). Also, the nonlinear term is written as $N(v) = \sum\limits_{i=0}^{\infty} H_i p^i$. Now, similar to the HPM, we have

$$\sum_{n=0}^{\infty} v_n p^n = u_0 + p\int_0^t \lambda(\tau)[R(\sum_{n=0}^{\infty} v_n p^n) + N(\sum_{n=0}^{\infty} v_n p^n) - g(\tau)]d\tau, \tag{2.33}$$

Finally by sorting coefficients with respect to powers of p, we have

$$\begin{aligned}
p^0 &: \quad v_0 = u_0 \\
p^1 &: \quad v_1 = \int_0^t \lambda(\tau)[R(v_0) + H_0(v_0) - g(\tau)]d\tau, \\
p^2 &: \quad v_2 = \int_0^t \lambda(\tau)[R(v_1) + H_1(v_0, v_1)]d\tau, \\
&\vdots \\
p^n &: \quad v_n = \int_0^t \lambda(\tau)[R(v_{n-1}) + H_{n-1}(v_0, v_1, \cdots, v_{n-1})]d\tau.
\end{aligned} \tag{2.34}$$

which is called the VHPM using **He's polynomials!!** For the selective zeroth approximation v_0 we used the initial values (2.54) . In the VHPM the initial approximation v_0 has been selected as

$$v_0(t) = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k. \tag{2.35}$$

2.2 Comparison between the ADM, the HPM, and the VHPM for solving Eq.

Those methods assumed the solution of (2.1) as a infinite series.
The components of those series will be computed by using iterative formula.

Theorem 2.2 *The He's polynomials* (2.28) *are the Adomian's polynomials* (2.20).

proof.see [39]

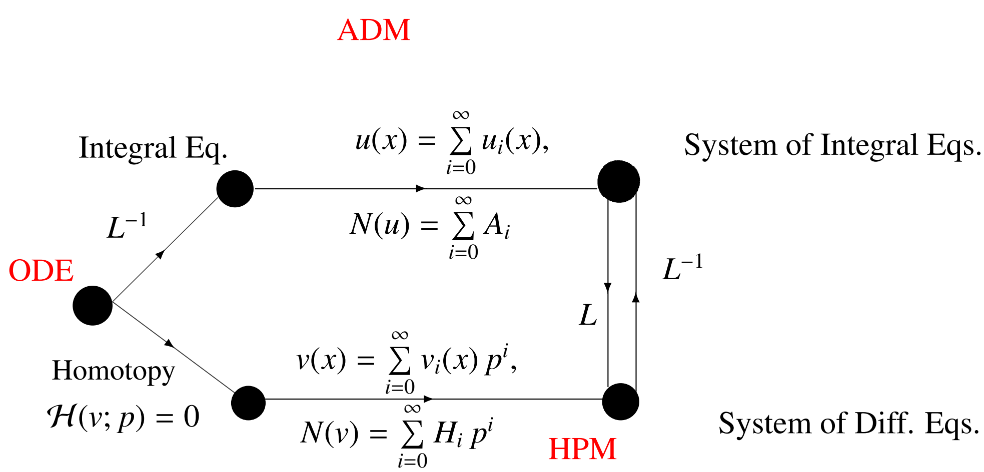
Theorem 2.3 *The HPM for solving Eq.* (2.1) *is equivalent the ADM when the homotopy* $\mathcal{H}(v;p)$ *is considered as* (2.23).

proof. Applying L^{-1} on both side of (2.29) we have

$$\begin{aligned}
v_0 &= \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k, \qquad (2.36)\\
v_1 &= -L^{-1}R[v_0] - L^{-1}H_0 + L^{-1}g(t),\\
v_2 &= -L^{-1}R[v_1] - L^{-1}H_1,\\
&\vdots\\
v_n &= -L^{-1}R[v_{n-1}] - L^{-1}H_{n-1}.
\end{aligned}$$

According to Theorem 1 we have $H_n = A_n$. In view of (2.21) and (2.25) we have

$$\begin{aligned}
\lim_{p\to 1} v = \lim_{p\to 1} \sum_{i=0}^{\infty} v_i p^i &= \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k + L^{-1}[g(\tau)] - L^{-1}R(v_0) - L^{-1}A_0 - \cdots\\
&= u_0 + u_1 + \cdots = \sum_{i=0}^{\infty} u_i = u
\end{aligned}$$



Theorem 2.4 *If we consider the homotopy $\mathcal{H}(v;p)$ as* (2.33) *for the VHPM. Then the VHPM is equivalent the ADM.*

proof. Substituting (2.25) and (2.27) into (2.33), we have

$$\begin{aligned}
&\mathcal{H}(v;p) = \sum_{n=0}^{\infty} v_n p^n - u_0 - p\int_0^t \lambda(\tau)[R(\sum_{n=0}^{\infty} v_n p^n) + N(\sum_{n=0}^{\infty} v_n p^n) - g(\tau)]d\tau = 0 \\
&\Rightarrow v_0 - u_0 + p[v_1 - \int_0^t \lambda(\tau)[R(v_0) + H(v_0) - g(\tau)]d\tau] \\
&- \int_0^t \lambda(\tau)[R(v_n) + H_n - v_{n+1}]p^{n+1} d\tau = 0,
\end{aligned}$$

By arranging the above equation according to the powers of p, we have

$$\begin{aligned}
&p^0: \quad v_0 - u_0 = 0, \\
&p^1: \quad v_1 - \int_0^t \lambda(\tau)[R(v_0) + H_0(v_0) - g(\tau)]d\tau = 0, \\
&p^2: \quad v_2 - \int_0^t \lambda(\tau)[R(v_1) + H_1(v_0, v_1)]d\tau = 0, \\
&\vdots \\
&p^{n+1}: \quad v_{n+1} - \int_0^t \lambda(\tau)[R(v_n) + H_n(v_0, v_1, \cdots, v_n)]d\tau = 0, \quad n = 0, 1, 2, \cdots.
\end{aligned} \tag{2.37}$$

From (2.37) we have

$$\begin{aligned}
v_0 &= u_0, \\
v_1 &= \int_0^t \lambda(\tau)[R(v_0) + H_0(v_0) - g(\tau)]d\tau, \\
v_2 &= \int_0^t \lambda(\tau)[R(v_1) + H_1(v_0, v_1)]d\tau, \\
&\vdots \\
v_{n+1} &= \int_0^t \lambda(\tau)[R(v_n) + H_n(v_0, v_1, \cdots, v_n)]d\tau, \quad n = 0, 1, 2, \cdots.
\end{aligned} \tag{2.38}$$

According to Theorem 1 we have $H_n = A_n$. In view of (2.31) and (2.35), substituting (2.38) into (2.32) leads us to

$$\begin{aligned} \nu &= \nu_0 + \nu_1 p + \nu_2 p^2 + \cdots \qquad (2.39)\\ \nu &= \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k + (\int_0^t \lambda(\tau)[R(\nu_0) + A_0 - g(\tau)]d\tau)p \\ &+ (\int_0^t \lambda(\tau)[R(\nu_1) + A_1]d\tau)p^2 + \cdots , \end{aligned}$$

so

$$\begin{aligned} \lim_{p\to 1} \nu &= \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k - \int_0^t \lambda(\tau)g(\tau)d\tau \qquad (2.40)\\ &+ (\int_0^t \lambda(\tau)[R(\nu_0) + A_0]d\tau) + (\int_0^t \lambda(\tau)[R(\nu_1) + A_1]d\tau) + \cdots . \end{aligned}$$

In [40], the first author proves that

$$\int_0^t \lambda(\tau)[.]d\tau = -L^{-1}[.]. \qquad (2.41)$$

Substituting (2.41) in (2.40) we have

$$\lim_{p\to 1} \nu = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k - L^{-1}[g(\tau)] - L^{-1}[R(\nu_0) + A_0] - L^{-1}[R(\nu_1) + A_1] + \cdots ,$$

hence

$$\lim_{p\to 1} \nu = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k - L^{-1}[g(\tau)] - L^{-1}[\sum_{i=0}^{\infty} R(\nu_i)] - L^{-1}[\sum_{i=0}^{\infty} A_i] = u.$$

So, we prove that $\lim_{p\to 1} \nu = u.$. In similar way we can prove that $u = \lim_{p\to 1} \nu$. In view of Theorems 2 and 3 we have:

Theorem 2.5 *Let* λ *is* (2.31) *and the homotopy* $\mathcal{H}(\nu; \mathrm{p})$ *is considered by* (2.33). *Then the VHPM for solving equation* (2.1) *is equivalent the HPM.*

proof. From (2.41) we have $\int_0^t \lambda(\tau)[.]d\tau = -L^{-1}[.]$ and substituting it in (2.33) we have

$$\sum_{n=0}^{\infty} \nu_n p^n = u_0 - pL^{-1}[R(\sum_{n=0}^{\infty} \nu_n p^n) + N(\sum_{n=0}^{\infty} \nu_n p^n) - g(\tau)], \qquad (2.42)$$

take limit of (2.42) when $p \to 1$ so we have

$$u = u_0 + L^{-1}(g(t)) - L^{-1}(R[u(t)]) - L^{-1}(N[u(t)]). \qquad (2.43)$$

Eq. (2.43) equivalent Eq. (2.55). That means variational homotopy perturbation method for equation (2.1) is same the HPM for (2.55).

Example 2.1 *Consider the following type of nonlinear differential equation*

$$\begin{aligned} u'(t) &= 1 + u^2(t)dt, \\ u(0) &= 0 \end{aligned} \tag{2.44}$$

here $L = \frac{d}{dt}[.]$ so $L^{-1}[.] = \int_0^t [.]d\tau$

The **ADM**	The **HPM**
$u(t) = t + \int_0^t u^2(x)dx$	$\mathcal{H}(v;p) = v'(t) - 1 - p\,v^2(t) = 0,$
$u(t) = \sum_{i=0}^{\infty} u_i(t),\ u^2(x) = \sum_{i=0}^{\infty} A_i$	$v(t) = \sum_{i=0}^{\infty} v_i(t)p^i, v^2(t) = \sum_{i=0}^{\infty} H_i p^i$
$\sum_{i=0}^{\infty} u_i(t) = t + \int_0^t \sum_{i=0}^{\infty} A_i dx$	$\sum_{i=0}^{\infty} v_i'(t)p^i - t - p\sum_{i=0}^{\infty} H_i p^i = 0$
$u_0(t) = t,$ $u_1(t) = \int_0^t A_0 dx$ $\vdots$ $u_n(t) = \int_0^t A_{n-1}dx,$	$p^0 :\ v_0'(t) - 1 = 0, v_0(0) = 0$ $p^1 :\ v_1'(t) - H_0 = 0, v_1(0) = 0$ $\vdots$ $p^n :\ v_n'(t) - H_{n-1} = 0, v_n(0) = 0$

$AdomianPolynomials$	$He'spolynomials$
$A_0 = u_0^2$	$H_0 = v_0^2$
$A_1 = 2u_0u_1$	$H_1 = 2v_0v_1$
$A_2 = u_1^2 + 2u_0u_2$	$H_2 = v_1^2 + 2v_0v_2$
$A_3 = 2u_1u_2 + 2u_0u_3$	$H_3 = 2v_1v_2 + 2v_0v_3$
$A_4 = u_2^2 + 2u_1u_3 + 2u_0u_4$	$H_4 = v_2^2 + 2v_1v_3 + 2v_0v_4$

The few terms of solution are:

ADM	HPM
$u_1(t) = \frac{t^3}{3}$	$v_1 = \frac{t^3}{3}$
$u_2(t) = \frac{2t^5}{15}$	$v_2 = \frac{2t^5}{15}$
$u_3(t) = \frac{17t^7}{315}$	$v_3 = \frac{17t^7}{315}$
$u_4(t) = \frac{62t^9}{2835}$	$v_4 == \frac{62t^9}{2835}$
$\vdots$	$\vdots$

So

$$u = \sum u_i = \lim_{p \to 1} \sum v_i p^i = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \cdots = \tan t$$

$$u(t) = t + \int_0^t u^2(x)dx,$$

To solve (2.45), by the VHPM

a correction functional for (2.44) is constructed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{Lu_n(\tau) - N\tilde{u_n}(\tau) - 1\}d\tau, \quad n \geq 0, \tag{2.45}$$

In view of (2.31) we find $\lambda = -1$ now according to VHPM we use following homotopy

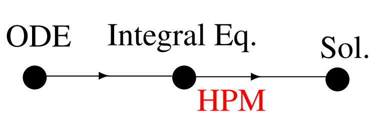
$$\sum_{n=0}^{\infty} v_n p^n = u_0 + p\int_0^t (-1)[-N(\sum_{n=0}^{\infty} v_n p^n) - 1]d\tau, \tag{2.46}$$

In view of initial condition we can rewrite (2.46) as

$$\sum_{n=0}^{\infty} v_n p^n = p(t + \int_0^t \sum_{n=0}^{\infty} H_n p^n)d\tau, \tag{2.47}$$

Finally by sorting coefficients with respect to powers of p, we calculate v_n $n = 0, 1, 2, \cdots$

If $p \to 1$ in (2.47) we have the ADM and it is equivalent applying the HPM for integral equation (2.45) .



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On comparison between iterative methods for solving nonlinear optimal control problems

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2.3 A brief view of tanh method, $(\frac{G'}{G})$-expansion Method and Simplest equation method

Suppose we have a nonlinear partial differential equation for $u(x,t)$ in the form

$$N(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0 \tag{2.48}$$

where $u(x,t)$ unknown function and dependent to x, t variables and N is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The transformation $u(x,t) = u(\xi)$, $\xi = k(x - c\,t)$ reduces Eq.(2.48) to the ordinary differential equation (ODE) as follow:

$$P(u, u', u'', u''', ...) = 0. \tag{2.49}$$

where $u = u(\xi)$ and prime denotes the derivative with respect to ξ , and k and c are constants. Exact solution of this equation can constructed as finite series

$$u(\xi) = \sum_{i=0}^{N} A_i\, Y^i, \quad A_N \neq 0, \tag{2.50}$$

where Y is $\tanh(m(\xi - \xi_0))$ in the tanh method , $Y = G(\xi)$ is a solution of the some ordinary differential equation referred to as the simplest equation in the simplest equation method and $Y = \frac{G'(\xi)}{G(\xi)}$ in the $(\frac{G'}{G})$-Expansion Method which is satisfies the following second-order linear ODE

$$G'' + \lambda G' + \mu G = 0. \tag{2.51}$$

Now $u(\xi)$ can be determined explicitly by using the following three steps:

- Step (1). By considering the homogeneous balance between the highest nonlinear terms and the highest order derivatives of $u(\xi)$ in Eq.(2.49), the positive integer N in (2.50) is determined.

- Step (2). By substituting (2.50) into (2.49) and collecting all terms with the same powers of Y together, the left hand side of Eq.(2.49) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of $A_i\,(i = 0, 1, 2, ..., n)$ and constants of $c,\ k\,(and\ \lambda, \mu$ in the $(\frac{G'}{G})$-Expansion method).

- Step (3). Solving the system of algebraic equations and then substituting the results into (2.50), gives solutions of (2.49).

2.3.1 Description of the tanh method

The tanh method for finding exact solutions of nonlinear differential equations was introduced more than 20 years ago. We should note some old publications of the application of the tanh method to look for exact solutions of nonlinear differential equations . Description of the tanh method can be found in papers [54,55]. The essence of this approach is as follows:

Let we have a partial differential equation as Eq.(2.48) and by transformation $\xi = k(x - c\,t)$ this equation is reduced to (2.49). Exact solution of this equation can constructed as finite series

$$u(\xi) = \sum_{i=0}^{N} A_i \tanh^i(m(\xi - \xi_0)) \tag{2.52}$$

where N is integer , coefficients A_i and parameter m are unknown values that can be found by using the step(1)- step(3).

2.3.2 Description of the simplest equation method

The simplest equation method is a very powerful mathematical technique for finding exact solutions of nonlinear ordinary differential equations(ODEs). It has been developed by Kudryashov [51] and used successfully by many authors for finding exact solutions of ODEs as well as PDEs. The first idea is to apply the simplest nonlinear differential

equations (the Riccati equation, the equation for the Jacobi elliptic faction, the equation for the Weierstrass elliptic function and so on) that have lesser order then the equation studied.

Let we have a partial differential equation as Eq.(2.48) and by transformation $\xi = k(x - c\,t)$ this equation is reduced to (2.49). Exact solution of this equation can constructed as finite series

$$u(\xi) = \sum_{i=0}^{N} A_i\,(G(\xi))^i \tag{2.53}$$

where $G(\xi)$ is a solution of some ordinary differential equation referred to as the simplest equation, and $A_0, A_1, A_2, ..., A_N$ are constants to be determined after substituting (2.53) into (2.49) as same as above.

The simplest equation has two properties:

(i) The order of simplest equation is lesser than equation (2.49).

(ii) we know the general solution(s) of the simplest equation or we know at least exact analytical particular solution(s) of the simplest equation.

The positive number N can be determined by step (1).

2.3.3 Description of the $(\frac{G'}{G})$-Expansion method

In this subsection we recall the basic idea of the $(\frac{G'}{G})$-expansion method [49,50]. Same as above, equation (2.48) by transformation $\xi = k(x - c\,t)$ is reduced to (2.49).

The $(\frac{G'}{G})$-expansion method is based on the assumption that the travelling wave solution of Eq.(2.49) can be expressed by a polynomial in $(\frac{G'}{G})$ as:

$$u(\xi) = \sum_{i=0}^{N} A_i \left(\frac{G'}{G}\right)^i, \quad A_N \neq 0, \tag{2.54}$$

where N is integer , coefficients $A_i (i = 1, 2, ..., N), \lambda, \mu$ are unknown values that can be found by using the step(1)–step(3). And $G = G(\xi)$ satisfies the second-order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{2.55}$$

here by using the general solutions of (2.55) we have

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + c_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi + c_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{c_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}\xi - c_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}\xi + c_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \end{cases} \tag{2.56}$$

and c_1 and c_2 are arbitrary constants.

2.4 Relations between of the tanh method, $(\frac{G'}{G})$-expansion method and simplest equation method

In this section we illustrate relations between these methods that sometimes these turn to other methods. Here we prove two theorem to show these cases.

Theorem 2.6 *The $(\frac{G'}{G})$-Expansion Method is a special case of the Simplest equation method when we use the Riccati equation as a simple equation.*

Proof 1 *with assuming*

$$Y(\xi) = \frac{G'(\xi)}{G(\xi)} \tag{2.57}$$

therefore Eq.(2.54) *turns into:*

$$u(\xi) = \sum_{i=0}^{N} A_i\, Y^i(\xi), \quad A_N \neq 0,\ A_i = const. \tag{2.58}$$

with using (2.57) *we have:*

$$G' = YG \Rightarrow G'' = Y'G + YG' \tag{2.59}$$

with using (2.55) *and* (2.59) *we have:*

$$Y_\xi = -Y^2 - \lambda Y - \mu \tag{2.60}$$

that this is the Riccati equation, where λ and μ are constants.

Therefore $(\frac{G'}{G})$-expansion method is equivalent with the simplest equation method when we use the Riccati equation as the simplest equation in this method.

Theorem 2.7 *If we use the Riccati or Bernoulli equations as the simplest equations in the simplest equation method then this method is equivalent with the tanh method.*

Proof 2 *The solution of the Riccati equation* (2.60) *is*

$$Y(\xi) = \sqrt{\mu + \frac{\lambda^2}{4}}\tanh(\sqrt{\mu + \frac{\lambda^2}{4}}(\xi - \xi_0)) + \frac{\lambda}{2} \tag{2.61}$$

substituting (2.61) *into* (2.53) *we have:*

$$\begin{aligned} u(\xi) &= \sum_{i=0}^{N} A_i\{\sqrt{\mu + \frac{\lambda^2}{4}}\tanh(\sqrt{\mu + \frac{\lambda^2}{4}}(\xi - \xi_0)) + \frac{\lambda}{2}\}^i \\ &= \sum_{i=0}^{N} A_i \sum_{j=0}^{i} \binom{i}{j}(\frac{\lambda}{2})^{i-j}(\sqrt{\mu + \frac{\lambda^2}{4}}\tanh(\sqrt{\mu + \frac{\lambda^2}{4}}(\xi - \xi_0))^j \qquad (2.62) \\ &= \sum_{i=0}^{N} b_i \tanh^i(m(\xi - \xi_0)) \qquad (2.63) \end{aligned}$$

from Eq.(2.63) *we can find coefficients b_i and parameter m. Therefore in this case simplest equation method is equivalent with the tanh method. Also with applying the Bernoulli equation:*

$$Y_\xi = aY(\xi) - Y^2(\xi) \tag{2.64}$$

the solutions of this equation are:

$$Y(\xi) = \begin{cases} \frac{a}{2}\left[1 + \tanh(\frac{a}{2}(\xi - \xi_0))\right], & a > 0, \\ \\ \frac{a}{2}\left[1 - \tanh(\frac{a}{2}(\xi - \xi_0))\right], & a < 0, \end{cases} \tag{2.65}$$

with substituting (2.65) *into* (2.53) *we have*

$$\begin{aligned} u(\xi) &= \sum_{i=0}^{N} A_i \left[\frac{a}{2}(1 \pm \tanh(\frac{a}{2}(\xi - \xi_0))\right]^i \\ &= \sum_{i=0}^{N} A_i \,(\frac{a}{2})^i \sum_{j=0}^{i} \binom{i}{j} (\pm 1)^j \tanh^j(\frac{a}{2}(\xi - \xi_0)) \end{aligned} \tag{2.66}$$

$$\Rightarrow u(\xi) = \sum_{i=0}^{N} b_i \tanh^i(m(\xi - \xi_0)) \tag{2.67}$$

we can find coefficients b_i and parameter m.

Therefore in this case too simplest equation method is equivalent with the tanh method.

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